14. Rabinovich, M.I., Method of averaging over stationary waves. Izv. Vyssh. Uchebn. Zavedenii, Radiofizika Vol.10, №2, 1967.
15. Kulikov, N.K., Conditions for the existence and determination of parameters of periodic motions of autonomous systems. Izv. Akad. Nauk SSSR, Mekhanika i Mashinostroenie, №4, 1959.
16. Malkin, I.G., Some Problems in the Theory of Nonlinear Oscillations. M., Gostekhizdat, 1956.
17. Mitropol'skii, Iu. A., Problems of the Asymptotic Theory of Nonstationary Oscillations. 2nd Ed. M., "Nauka", 1964.
18. Gantmakher, F.P., Matrix Theory. 2nd ed., M., "Nauka", 1966.
19. Courant, R., Partial Differential Equations, New York, Interscience, 1962.
20. Karpman, V.I. and Krushkal', E.M., Modulated waves in nonlinear dispersive media. JETP Vol. 55, №2, 1968.
21. Tam, G.K.W., Nonlinear dispersion of cold plasma waves. J. Plasma Phys. Vol. 4, №1, 1970.
22. Vaast, R., Variational principle for certain nonconservative systems. Amer. J. Phys. Vol. 35, №5, p.419, 1967.
23. Kas'ianov, V.A. and Tkachenko, N.E., Description of dissipative systems with the aid of Hamiltonian formalism. Prikl. Mekh. Vol. 6, №2, 1970.

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ON NONLINEAR TRANSVERSE RESONANT OSCILLATIONS IN AN ELASTIC LAYER AND A LAYER OF PERFECTLY CONDUCTING FLUID

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One proves the equivalence of the equations of the one-dimensional plane motion of an isotropic nonlinearly-elastic body and that of a perfectly conducting compressible fluid moving in an external magnetic field, the magnetic permeability of the fluid being an arbitrary function of the density and of the modulus of intensity of the magnetic field. For these models of the continuous medium one considers essentially the nonlinear problem of the transverse oscillations induced in an infinite layer by the periodic action of external tangential forces at one of the plane boundaries, while at the other one a perfect reflection of the waves is assumed. The singularity of the bahavior of the forced resonant oscillations are developed in the case when in the elastic body the velocity of the longitudinal waves is much larger than the velocity of the transverse waves and in the fluid the velocity of the sound exceeds by far the velocity of the Alfvén waves. One establishes the relation between the amplitude of the constraining forces and

the degree of nearness of its frequency to the resonant frequency when weak shock waves appear in the layer.

1. Nonlinearly elastic layer. The equations of the dynamics of an isotropic elastic body for a one-dimensional motion with plane symmetry is written in the form

$$\rho_0 \frac{\partial^2 w_1}{\partial t^2} = \frac{\partial}{\partial \xi} p_{\parallel}, \quad \rho_0 \frac{\partial^2 w_2}{\partial t^2} = \frac{\partial}{\partial \xi} p_{\perp i} \qquad (i = 2, 3) \tag{1.1}$$

$$p_{\parallel} = \rho_0 \partial F / \partial s, \quad p_{\perp i} = \rho_0 h_i \partial F / \partial (h^2 / 2), \qquad s \equiv \partial w_1 / \partial \xi, \quad h_i \equiv \partial w_i / \partial \xi$$

Here w_1, w_2, w_3 are the components of the displacement vector of a Lagrange particle in the Cartesian system of observation coordinates x^1, x^2, x^3; ξ is the Lagrange coordinate which in the initial unloaded state $p_{\parallel} = p_i = 0$ for $s = h = 0$ coincides with the Cartesian coordinate; ρ_0 is the initial density; $F(s, h^2/2)$ is the isothermal density of the free energy.

The equations (1.1) are obtained from the general equations of the dynamics of an elastic body [1], which in a Cartesian system of reference have the form

$$\frac{\partial^2 w_i}{\partial t^2} = \frac{\partial}{\partial \xi^j} \left[\left(\delta_{ik} + \frac{\partial w_i}{\partial \xi^k} \right) \frac{\partial F}{\partial \varepsilon_{jk}} \right]$$

where ξ^1, ξ^2, ξ^3 are the Lagrange coordinates and ε_{jk} are the components of the strain tensor. In the derivation of (1.1) one takes into account that in the two-dimensional case the invariants of the strain tensor can be expressed in terms of s and h^2, and the nonzero components ε_{jk} have the form

$$\varepsilon_{11} = s + (s^2 + h^2) / 2, \qquad \varepsilon_{1k} = h_k / 2 \qquad (k = 2, 3)$$

In the following we restrict ourselves to the case $h_3 = 0$, $h_2 = h$. Then, from the system (1.1) for the displacements, we obtain the following equations for the longitudinal stress $p_{\parallel}$ and for the shearing stress $p_{\perp}$:

$$\frac{\partial}{\partial t} \left[\frac{F_{hh}}{F_{ss} a_{\perp}^2} \frac{\partial p_{\parallel}}{\partial t} - \frac{F_{sh}}{F_{ss} a_{\perp}^2} \frac{\partial p_{\perp}}{\partial t} \right] = \frac{\partial^2}{\partial \xi^2} p_{\parallel} \tag{1.2}$$

$$\frac{\partial}{\partial t} \left[- \frac{F_{sh}}{F_{ss} a_{\perp}^2} \frac{\partial p_{\parallel}}{\partial t} + \frac{1}{a_{\perp}^2} \frac{\partial p_{\perp}}{\partial t} \right] = \frac{\partial^2}{\partial \xi^2} p_{\perp}$$

$$F_{hh} \equiv \partial^2 F / (\partial h)^2, \qquad F_{hs} \equiv \partial^2 F / (\partial s \partial h), \qquad a_{\perp}^2 = F_{hh} - (F_{hs})^2 / F_{ss}$$

The equations of the characteristics of the systems (1.2), (1.1) (for $h_3 = 0$) are

$$\left(\frac{d\xi}{dt} \right)^2 = \frac{1}{2} \left[F_{ss} + F_{hh} \pm \sqrt{(F_{ss} - F_{hh})^2 + 4F_{sh}^2} \right] \tag{1.3}$$

Here $d\xi / dt$ is the propagation velocity of wave fronts of the deformations and stresses.

We assume that the elastic body is such that $F_{ss} \gg F_{sh} \gg F_{hh}$ in the domain of the values s and h, characteristic for the problems under investigation. Then it follows from (1.3) that the propagation velocity of the "fast longitudinal" waves is equal to $\sqrt{F_{ss}}$ and the velocity of the "slow transverse" waves is determined from the relation (1.2), where $a^2_{\perp} = dF(s, h) / dh$ along the curve $\partial F / \partial s = \text{const}$.

We consider a layer of an elastic material of thickness L, which rests without separation on an absolutely rigid foundation without tangential friction forces; on the upper boundary of the layer we have a periodic tangential force $A \sin \omega t$ and a constant normal load q_0. This corresponds to the boundary conditions for the system (1.1) or (1.2)

$$p_{\perp}=0, \qquad w_{1}=0 \quad \text{for } \xi=0$$
$$p_{\parallel}=q_0, \qquad p_{\perp}=A\sin\omega t \quad \text{for } \xi=L \tag{1.4}$$

We will look for a periodic solution of the formulated problem, for whose realization it is necessary that the layer should not accumulate energy, i.e. the work of the external tangential force over a period be equal to zero

$$\int\limits_{-\pi/\omega}^{\pi/\omega} A\sin\omega t\,\frac{\partial w_{\perp}}{\partial t}\,dt=0 \tag{1.5}$$

If the amplitude of the exciting force is sufficiently small, then the desired solution can be obtained from the solution of the linear problem with linearized boundary conditions. This is a stationary transverse wave

$$p_{\perp}=A\sin\omega t\,\frac{\sin(\omega\xi/a)}{\sin(\omega L/a)}, \qquad p_{\parallel}=q_0, \qquad a=a_{\perp}(h=0) \tag{1.6}$$

However, when the frequency ω approaches the resonance frequency $\omega_{*}=n\pi a/L$ ($n=1,2,\ldots$) then inside the layer one obtains infinite stresses. From here follows the essential nonlinearity of the formulated problem near the resonance even for a small amplitude of the exciting force.

In view of the fact that the problem has a characteristic length L and a chracteristic time $T=2\pi/\omega$, where $L/T\sim a_{\perp}\ll a_{\parallel}$, the solution can be sought in the form of an expansion with respect to two small parameters F_{hs}/F_{ss} and F_{hh}/F_{ss}.

In the first approximation the longitudinal stresses turn out to be constants; this corresponds to the absence of the radiation of longitudinal waves. The relations of the longitudinal and the transverse deformations turn out to be finite by virtue of the equality $p_{\parallel}(s,h)=q_0=\text{const}$. Expressing s in terms of h, we obtain $a_{\perp}^2=a_{\perp}^2(h)$.

From (1.1) we obtain the equation for the shearing displacement

$$\frac{\partial^2 w_{\perp}}{\partial t^2}-a_{\perp}^2\frac{\partial^2 w_{\perp}}{\partial \xi^2}=0 \tag{1.7}$$

From (1.2) we obtain the equation for the shear stress

$$\frac{\partial}{\partial t}\left[\frac{1}{a_{\perp}^2}\frac{\partial}{\partial t}\,p_{\perp}\right]=\frac{\partial^2}{\partial\xi^2}\,p_{\perp}$$

The Eq. (1.7) can be written in two equivalent forms

$$\left(\frac{\partial}{\partial t}\pm a_{\perp}\frac{\partial}{\partial\xi}\right)\left[\frac{\partial w_{\perp}}{\partial t}\mp\int\limits_0^h a_{\perp}(h)\,dh\right]=0$$

Hence it follows that

$$\frac{\partial w_{\perp}}{\partial t}\mp\int\limits_0^h a_{\perp}(h)\,dh=2a\,\varphi^{\pm}(c_{\pm}) \tag{1.8}$$

where c_{+} is found from the equation $\xi=\xi(t,c_{+})$, which determines the family of the integral curves of the equation $d\xi/dt=\pm a_{\perp}'(h)$.

In view of the periodicity of the desired solution, the functions φ^{-} and φ^{+} have to be periodic, if for the parameters of the families of characteristics we take the moment of the intersection of the corresponding characteristic with the line $\xi=0$. From the conditions (1.4) for $\xi=0$ it follows that

$$\varphi^-(c) = \varphi^+(c) = \varphi(c) \tag{1.9}$$

We investigate now the case of a weakly nonlinear material, for which the free energy is an analytic function of the invariants J_1, J_2, J_s [1] at the point $J_1 = J_2 = J_3 = 0$. It can be shown that in this case the function $a_\perp(h)$ can be expanded in a series with even exponents. We restrict ourselves to the first two approximations

$$a_\perp(h) \approx a(1 + 3\alpha h^2) \tag{1.10}$$

From the equations of the characteristics we have, to within terms $\sim h^4$

$$c_+ = \lambda + 3\alpha \int\limits_{\lambda}^{(\lambda+\mu)/2} [\varphi(\lambda) - \varphi(2\tau - \lambda)]^2 \, d\tau$$

$$c_- = \mu - 3\alpha \int\limits_{(\lambda+\mu)/2}^{\mu} [\varphi(\mu) - \varphi(2\tau - \mu)]^2 \, d\tau \tag{1.11}$$

$$\lambda = t - \xi / a, \quad \mu = t + \xi / a$$

From (1.8), taking into account (1.9), we obtain

$$h + \alpha h^3 = \varphi(c_-) - \varphi(c_+) \tag{1.12}$$

The function φ is found from the conditions (1.4) for $\xi = L$

$$\rho_0 a^2 (h + 2\alpha h^3) = A \sin \omega t$$

Substituting here the solution of (1.12) for $\xi = L$, we obtain for φ the functional equation

$$\rho_0 a^2 [\varphi(c_-) - \varphi(c_+) + \alpha(\varphi(\lambda) - \varphi(\mu))^3] = A \sin \omega t \tag{1.13}$$

For a small deviation of the frequency of the exciting force from the resonance frequency, we put in the nonlinear terms of (1.13) $\omega = \omega_*$, $\lambda = t - n\pi / \omega$, $\mu = t + h\pi / \omega$; we replace the difference $\varphi(c_-) - \varphi(c_+)$ by the expression

$$-\omega^{-1} 2n\pi d\varphi(t - n\pi(\omega)/dt \, [3\alpha\varphi^2(t - n\pi / \omega) + 3\alpha\Omega - (\omega - \omega_*) / \omega_*] \tag{1.14}$$

Here we have made use of the condition

$$\int\limits_{-\pi/\omega}^{+\pi/\omega} \varphi(t) \, dt = 0 \qquad \left(\Omega = \frac{\omega}{\pi} \int\limits_{-\pi/\omega}^{\pi/\omega} \varphi^2(t) \, dt\right) \tag{1.15}$$

and we have taken into account the periodicity of $\varphi(t)$.

We discard the last term in the left-hand side of (1.13) since it is proportional to $(\omega - \omega_*)^3$. Integrating (1.13) with respect to time, we obtain finally

$$\varphi^3(t) + \varphi(t)[3\Omega + \vartheta] + \nu \cos \omega t = 0$$

$$\vartheta = -(\alpha\omega_*)^{-1}(\omega - \omega_*), \quad \nu = (2n\pi\alpha\rho_0 a^2)^{-1}(-1)^n A \tag{1.16}$$

If at the boundary $\vartheta = L$ instead of the condition $p_{11} = q_0$ we put $w_1 = 0$, then instead of (1.16) we obtain the equation

$$\varphi^3(t) + \varphi(t)[\Omega + \vartheta] + \nu \cos \omega t = 0 \tag{1.17}$$

Thus, taking account of a small nonlinearity in the resonance leads to the solution (1.8), where the functions c_+ and c_- are given by the formulas (1.11).

Assume that the function φ has a finite discontinuity at some point ($\varphi(t)$ is a discontinuous function in some range of the frequency near the resonance, see Sect. 3). For the stability (evolution) of the discontinuity it is necessary that it catches up with the disturbances ahead and that it be caught up by disturbances behind the discontinuity. At such discontinuities, from the conservation of momentum it follows that

$$D^2 = a^2 [1 + \alpha (h^2_{(1)} + h_{(2)}h_1 + h^2_{(2)})]$$

where D is the velocity of the shock wave with respect to the particles, $h_{(1)}$ is the value of h ahead of the shock, $h_{(2)}$ is the value of h behind the shock.

In view of the fact that at the evolutionary discontinuities we have $a^2_{\perp(1)} < D^2 < a^2_{\perp(2)}$ at the weak shock waves we must have either $h_{(1)}/ h_{(2)} > 1$ or $h_{(1)} / h_{(2)} \leqslant -2$. At these discontinuities the longitudinal stress remains continuous, therefore we neglect the radiation of the longitudinal waves on the weak shearing shock waves.

Expressions (1. 8) may be linearized in zeroth approximation, while the function φ (t) may be determined as before from equations (1.16), (1.17). However in the linear range both shock waves and rarefaction waves of the nonlinear solution (1. 8) will result in discontinuous solutions. In order to distinguish the rarefaction waves from the shock waves in the linear approximation, it is necessary to varify the criterion of evolution: if this criterion is not satisfied, then the discontinuity represents the limiting form of the rarefaction wave, continuous in the nonlinear approach (1. 8) - (1.11). The nonlinear solution (1. 8) is nonunique in the vicinity of the evolutionary discontinuity. In this case the construction of a unique discontinuous solution is carried out with the aid of formulation of superfluous branches.

2. Layer of a perfectly conducting fluid [2]. The ponderomotive force in magnetizing media in the absence of polarization and space charge is equal to [1]

$$\frac{1}{c} (\mathbf{j}\times\mathbf{B}) \frac{1}{8\pi} (B_\alpha \nabla H^\alpha - H_\alpha \nabla B^\alpha)$$

Starting from this, one can show that the one-dimensional plane motion of a magnetizing compressible perfectly conducting fluid in an external magnetic field B_0, applied orthogonal to the plane of symmetry, can be described in Lagrange coordinates by the equations

$$\rho_0 \frac{\partial^2 w_1}{\partial t^2} = -\frac{\partial}{\partial \xi}\left(p + \frac{H_i B^i}{8\pi}\right) \qquad (i = 2, 3)$$
$$\rho_0 \frac{\partial^2}{\partial t^2}\left[B_i\left(1 + \frac{\partial w_1}{\partial \xi}\right)\right] = \frac{B_0^2}{4\pi\rho_0}\frac{\partial^2 H_i}{\partial \xi^2} \tag{2.1}$$

From the equations of heat flow in the reversible adiabatic case it follows

$$du = \frac{\mathbf{H}}{4\pi}\, d\,\frac{\mathbf{B}}{\rho} - \left(p + \frac{\mathbf{HB}}{8\pi}\right) d\,\frac{1}{\rho} \tag{2.2}$$

where u is the internal energy function of the fluid including the energy of the magnetic field per unit of mass.

It follows from (2.2), (2.1) that if we introduce the potentials w_i $(i = 2, 3)$ for the new unknown functions $h_i \equiv B_i (1 + s) / B_0$: $h_i = dw_i / d\xi$, then the Eqs. (2.1) for w_1, w_2, w_3 will have the form (1.1) provided the function u depends on the components of the vector $\mathbf{B}$ through its modulus.

In the absence of the magnetization of the fluid, the system (2.1) is converted into the system of equations of the one-dimensional magnetohydrodynamics [3]. In this particular case the function u takes the form

$$u\,(s, h^2 / 2) = U\,(s) + h^2 B_0{}^2 (8\pi\rho_0 + 8\pi\rho_0 s)^{-1}$$

where $U\,(s)$ is the internal energy of the barotropic fluid.

In the given notation the equations of the characteristics of the system (2.1) are (1.3). We assume that the velocity of sound in the weakly compressible fluid exceeds by far the velocity of the transverse Alfvén waves $a_\perp^2$ (in the presence of magnetization $a_\perp^2 \neq B^2/4\pi\rho_0$).

In this case the boundary conditions (1.5) acquire another interpretation: the layer of the described fluid is enclosed between two infinite conducting planes. One of the planes is fixed and de-energized and on the other a periodic current is supplied and a constant normal load is applied. If the frequency of the current approaches the resonant frequency, then even for small amplitudes of the current, the problem becomes nonlinear. Due to the method presented in Sect. 1, in the case of a weak nonlinearity, the determination of the solution reduces again to the analysis of the algebraic equation (1.16).

If instead of the transverse oscillations on the boundary $\xi = L$ only longitudinal oscillations are stimulated according to a harmonic law, then in the case of a weak nonlinearity the determination of the periodic solution of this gas dynamics problem ($B_i = 0$) at resonance, reduces, as it has been shown by Chester [4], to the analysis of a quadratic equation.

3. Investigation of equation (1.16). It is convenient to write Eq. (1.16) in the form

$$y^3 + 3\,\mathrm{sign}\,(3\,\Omega + \zeta)\,y + 2y \cos \omega t = 0 \qquad (3.1)$$
$$y \equiv |\,3\Omega + \zeta\,|^{-1/2} \sqrt{3}\varphi, \qquad \gamma = 2\,|\,\Omega + \zeta\,/\,3|^{-1/2}\,\nu$$

It is necessary to distinguish three cases

1°. $3\Omega + \zeta < 0$, $|\,\gamma\,| \leqslant 1$; 2°. $3\Omega + \zeta < 0$, $|\,\gamma\,| > 1$; 3°. $3\Omega + \zeta > 0$

Case 1. Equation (3.1) for $|\,\gamma\,| < 1$ determines three continuous functions

$$y_k\,(t) = -\,2\,\mathrm{sign}\,\gamma \cos\,[{}^1/_3 \arccos\,(\,|\gamma| \cos \omega t) + 2\pi k\ /\ 3] \qquad (k = 1, 2, 3)$$

From the three solutions only $y_2\,(t)$ satisfies condition (1.15). For the determination of Ω we have in this case the finite relation

$$\Omega = (3\pi)^{-1}\,(4\zeta + 12\Omega) \int\limits_0^\pi \cos^2\,[{}^1/_3 \arccos\,(|\,\gamma\,| \cos \theta) + 4\pi/3]\, d\theta$$

From the smooth pieces of the continuous solutions $y_1\,(t)$ and $y_3\,(t)$ one can construct uniquely discontinuous solutions satisfying the condition (1.15) and having a minimum number of discontinuities, by enlisting additional conditions of evenness of the discontinuous solution (then the condition (1.6) of no energy, supplied to the system over a period, is automatically satisfied)

$$y_a = \begin{cases} y_1\,(t) \\ y_3\,(t) \end{cases} \quad \text{or} \quad y_b = \begin{cases} y_3\,(t) & \text{for } -\pi\,/\,2 < \omega t < \pi\,/\,2 \\ y_1\,(t) & \text{for } \pi\,/\,2 < \omega t < 3\pi\,/\,2 \end{cases} \qquad (3.2)$$

. the first discontinuous solution. Ω satisfies the relation

$$3\pi\Omega = -(\zeta + 3\Omega) \int\limits_{-\pi/2\omega}^{\pi/2\omega} y_1^2(t)\, dt$$

For the second

$$3\pi\Omega = -(\zeta + 3\Omega) \int\limits_{-\pi/2\omega}^{\pi/2\omega} y_3^2(t)\, dt$$

For $|\gamma| \ll 1$ the functions $y_k(t)$ have the asymptotics

$$y_1(t) = \sqrt{3} - \gamma \cos \frac{\omega t}{6}, \quad y_0(t) = \gamma \cos \frac{\omega t}{3}, \quad y_3(t) = -\sqrt{3} - \gamma \cos \frac{\omega t}{6}$$

This means that the solution $y_2(t)$ tends to the ordinary linear solution. The discontinuous solutions tend to periodic solutions of rectangular form. The discontinuities are transfered with velocity a.

If $|\gamma| = 1$, we obtain

$$y_k(t) = -2 \operatorname{sign} \gamma \cos [{}^1/_3 (\omega t + 2\pi k)] \quad (k = 1, 2, 3) \tag{3.3}$$

The continuous solution is constructed from the various functions $y_k(t)$ and it has a discontinuity of the derivative at $\omega t = \pi m$ (Fig. 1, curve y_b).

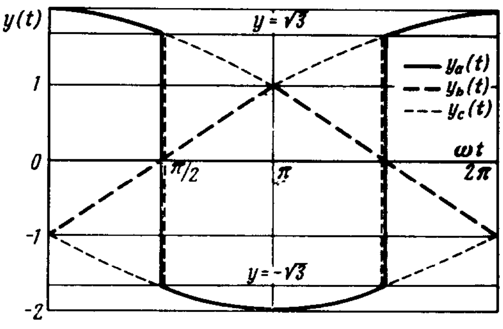

Fig. 1.

For the continuous solution

$$\nu = 2\,[(9\pi - 9\sqrt{3})^{-1} \pi\, |\zeta|]^{3/2}$$

$$\Omega = \frac{3\sqrt{3} - 2\pi}{9(\pi - \sqrt{3})}\, \zeta$$

For one of the discontinuous solutions (Fig. 1, curve y_a) the relation between the amplitude of the external perturbation and the deviation of the frequency from the resonance frequency has the form

$$\nu = 2\,[(9\pi + 9\sqrt{3})^{-1} \pi\, |\zeta|]^{3/2}$$

The other discontinuous solution, in addition to the finite discontinuities for $\omega t = \pi/2 + \pi m$ has the weak discontinuities for $\omega t = \pi m$ $(m = 0, \pm 1, \ldots)$ (Fig. 1, curve y_c).

C a s e 2. Equation (3.1) determines one periodic real multiple-valued function $y(t)$. Its form is represented in Fig. 2.

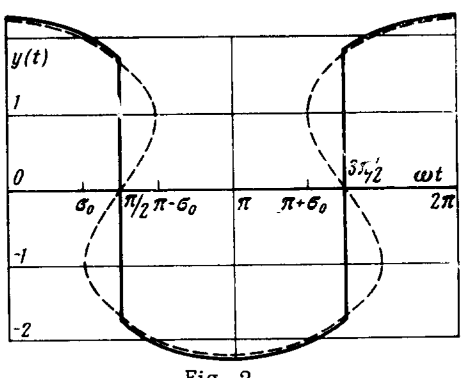

Fig. 2.

Since $y(t)$ is multiple-valued, there is no continuous solution and the discontinuous solution is given by the formulas

$$y(t) = \begin{cases} -2 \operatorname{sign} \gamma \operatorname{ch} \sigma_1, & -\sigma_0 < \omega t \leqslant \sigma_0 \\ -2 \operatorname{sign} \gamma \cos \sigma_2, & \sigma_0 < \omega t < \pi/2 \\ -2 \operatorname{sign} \gamma \cos (\sigma_2 + 2\pi/3), & -\pi/2 < \omega t \leqslant \pi - \sigma_0 \\ 2 \operatorname{sign} \gamma \operatorname{ch} \sigma_1, & \pi - \sigma_0 < \omega t \leqslant \pi + \sigma_0 \end{cases} \tag{3.4}$$

$\delta_0 = \arccos (\gamma)^{-1}$, $\delta_1 = {}^1/_3 \operatorname{arcch} (\gamma \cos \omega t)$ $\delta_2 = {}^1/_3 \operatorname{arcos} (\gamma \cos \omega t)$

Case 3. The solution of (3.1) is unique, single-valued and continuous

$$y(t) = -2 \operatorname{sh} [{}^1/_3 \operatorname{arcsh} (\gamma \cos \omega t)] \tag{3.5}$$

In particular, for $|\gamma| \to \infty$ from (1.16) we obtain

$$\varphi(t) = -\sqrt[3]{\nu \cos \omega t} \tag{3.6}$$

In this case

$$\Omega = \pi^{-1}\nu^{2/3} \int_0^\pi \cos^{2/3} \theta d\theta, \qquad \zeta = -3\Omega$$

For the resonance $\zeta = 0$, and for γ we obtain the equation

$$\frac{\pi}{4} = \int_0^\pi \operatorname{sh}^2 \left\{\frac{1}{3} [\operatorname{arcsh}(\gamma \cos \theta)]\right\} d\theta$$

If $|\gamma| \ll 1$, then solution (3.5) reduces to the linear solution.

4. Physical interpretation of the results. In the nonlinear theory near resonance, the amplitudes of the oscillations of the shear deformations and the strength of the magnetic field have, relative to the magnitude of the external field B_0, an order equal to $\nu^{1/3} = \text{const} [A (\alpha\rho_0 a^2)^{-1}]^{1/3}$. The transverse oscillations cause the appearance of longitudinal oscillations.

We assume that the amplitude of the exciting force is fixed. We follow the evolution of the shape of the oscillations with the variation of the frequency. Let us assume that ω decreases monotonically. For $\omega > \omega_*$ a nonlinear smooth oscillation takes place in the layer, described by (3.5). Oscillations without discontinuities will occur also in the case of resonance $\omega = \omega_*$ and for $\omega < \omega_*$ down to the frequency ω^a

$$\omega^a = \omega_* - \beta^a l \ (l = [\alpha\pi n A^2(\rho^2{}_0 a L^3)^{-1}]^{1/3})$$

The values of the constant β are given below for different boundary conditions at $\xi = L$: in the upper line for the condition $p_{11} = q_0$; in the lower line for $w_1 = 0$. The formulas hold under the condition $l \ll 1$.

β^a	β^c	β^d	β^b
$3\cdot 2^{-4/3}\pi^{-1}\int_0^\pi \cos^{2/3}\theta d\theta$	$9(4\pi)^{-1}(\pi - \sqrt{3})$	$9/4$	$9(4\pi)^{-1}(\pi + \sqrt{3})$
$2^{-4/3}\pi^{-1}\int_0^\pi \cos^{2/3}\theta d\theta$	$(5\pi - 3\sqrt{3})(4\pi)^{-1}$	$5/4$	$5(4\pi)^{-1}(\pi + 3\sqrt{3})$

For the frequency ω^a (see (3.6)) the profile of the wave has a vertical tangent at the time $\omega^a t = \pi/2 + \pi m$ $(m + 0, \pm 1, ..)$. For the frequencies $\omega \leqslant \omega^a$ the motion in the layer can be accomplished with weak shock waves. Such oscillations are described by the solution (3.4) (Fig. 2) up to the frequency $\omega^b = \omega_* - \beta^b l$ and by formula (3.2) for y_a in the case $\omega < \omega_b$. For $\omega = \omega_b$ the form of this solution is represented in Fig. 1 (curve y_a).

For the frequency $\omega^c = \omega_* - \beta^c l$ $(\omega^c < \omega^a)$, in addition to the described solution, there emerges a continuous solution with weak discontinuities at $\omega^c t = \pi m$ (Fig. 1, curve y_b), i.e., a peculiar bifurcation of the solution takes place. For $\omega < \omega^c$ the continuous solution is described by the analytic function $y_2(t)$. In addition to these two solutions, for the frequency $\omega^d = \omega_* - \beta^d l$ there emerges a third discontinuous solution, having jumps at $\omega^d t = \pi / 2 + \pi m$ (Fig. 1, curve y_c). For frequencies smaller than ω_d, the weak discontinuities of this solution disappear and the solution is described by formula (3.2) for y_b.

Thus, the periodic solution of Eq. (1.7) with the boundary conditions (1.5) is unique for $\omega > \omega^c$; there are two solutions for $\omega^c \geqslant \omega > \omega^d$ (one is discontinuous, the other is continuous); three solutions (one continuous, two discontinuous) are possible for $\omega < \omega^a$

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BIBLIOGRAPHY

1. Sedov, L. I., Mechanics of the continuous medium, Vol. 1, 2. Moscow, "Nauka" 1970.
2. Chester, W. Resonant oscillations in closed tubes. J. Fluid. Mech., Vol. 18, pp. 44-64, 1964.
3. Sibgatullin, N. R., On resonant Alfvén waves in an infinite layer of weakly compressible fluid. Tez. dokl. nauchn. konfer. NII mekhaniki MGU. M., 1970.
4. Kulikovskii, A. G. and Liubimov, G. A., Magnetohydrodynamics, Moscow, Fizmatgiz, 1962.

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ON THE PRESSURE ON AN ELASTIC HALF-SPACE BY A WEDGE-SHAPED STAMP

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The problem of the effect of an absolutely rigid stamp with a wedge planform on an elastic space is considered. There is assumed to be no friction in the domain of contact between the stamp and the half-space.

Galin first considered this problem in [1]. The effect of the stamp on the half-space was accompanied, in that paper, by the effect of some loading outside it. A characteristic singularity of this solution is the fact that the contact pressures $p(\tau, \varphi)$ have a r^{-1} singularity at the wedge apex.

Later, Rvachev attempted to solve the mentioned problem without the outside loading [2]. He reduced it to an eigenvalue problem for a certain differential equation on a sphere and utilized the Galerkin method. The Rvachev solution has a $r^{\gamma-1}$ singularity at the wedge apex, where $0 < \gamma(\alpha) < 1$, and 2α is the wedge angle.

In this paper the problem of a wedge-shaped stamp with an arbitrary base is apparently successfully solved analytically for the first time by utilizing the